Quantification of Uncertainty Related to Stress Estimation in Turbine Engine Blades

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Outline

• Project overview
• Motivations
• Objectives and Approach
• Analysis of uncertainty propagation during stress estimation
• Introduction to Bayesian Nets:
  – Application to beam example
• Concluding Remarks
• Future Work
• 3-year project started in March 2005;

• Co-PIs combine expertise in structural dynamics, probabilistic, reliability;

• Interaction between GATech and AEDC
  – POCs: P. Cento, R. McAmis, T. Tibbals, S. Arnold;

• AEDC is providing guidance, information on current testing procedures and experimental data;

• Interaction is based on bi-weekly tele-cons held regularly since beginning of the project, and on-site meetings.
Motivations

• Satisfactory assessment of stress levels in turbine disks is key to engine reliability

• It relies on:
  – high-fidelity analysis tools
  – high measurement accuracy
  – identification of uncertainties
  – sensitivity assessment

• A probabilistic methodology is being developed to:
  – improve stress prediction
  – support analysis and interpretation of test results
Objectives and approach

• Objectives:
  – Evaluate uncertainty in stress estimation in turbine blades;
  – Apply statistical techniques to combine information from FE models and test results.

• Approach:
  – Identify sources of uncertainty;
  – Propagate uncertainty through Monte Carlo simulations;
  – Employ Bayesian nets to combine modeling assumptions with evidence obtained from tests;
  – Update uncertainty estimations, and reduce discrepancies between physical system and model.
Approach

• General approach to uncertainty estimation:

Uni-directional SISO information flow (Monte Carlo, FORM/SORM, etc.)

• More complex situation.

Bayesian nets are proposed as a tool to handle additional information from tests
Analysis and propagation of uncertainty during the stress estimation
Model-based Stress Inference Process

- Evaluation of maximum stresses relies on model information.
  - Identification of the mode correlating to the maximum sensor amplitudes (forced response) defines the modal participation factors;
  - Experimental data and identified modal information are used to extrapolate maximum stress values.

Example: two-sensor case
Model-based Stress Inference Process

Given \( a_{s,j,\text{max}} \) (i.e. the maximum amplitude measured by the j-th sensor), the maximum response in the structure can be extrapolated as follows:

STEP 1: Minimization of the fitness parameter.

\[
\min_{s,i} \left\{ \left| f_s - f_i \right| + \left| \frac{a_1(f_s)}{a_2(f_s)} - \frac{\phi_i(x_1)}{\phi_i(x_2)} \right| \right\}, \quad i = 1, 2, \ldots, n_m
\]

\( s = 1, 2 \)

STEP 2: Maximum-value estimate via scaling of modal responses \( \Phi_p \) and \( \Phi_q \):

\[
a_{\text{max}} = \max \left\{ a_{s,1} \left( f_{\text{max}, p} \right) \frac{\max(\phi_p)}{\phi_{s,1, p}}, a_{s,2} \left( f_{\text{max}, q} \right) \frac{\max(\phi_q)}{\phi_{s,2, q}} \right\}
\]
System Response Estimation under Uncertainty

Modal Analysis on system model

Data from (simulated) experiments on system (model)

Modeling Uncertainty
(via mode-shape perturbation)

Sensor-based Uncertainty
(normally distributed error added to nominal values)

Prediction Scheme
(two-sensor case)

1. Identification of modes $\Phi_p$ and $\Phi_q$
2. Maximum-response estimation via modes $\Phi_p$ and $\Phi_q$
Sources of uncertainty

• Material and geometric uncertainties;
• Sensor related uncertainties:
  – sensor measurement uncertainty:
    \[
    \hat{\alpha}_s(f) = a_s(f) + A \times N(0, \sigma^2), \quad s = 1, 2
    \]
    \[A = \text{magnitude of sensor “noise”}\]
  – sensor placement;
• Modeling uncertainties:
  – Model has limited fidelity, i.e. it does not reproduce component behavior exactly.
Modeling uncertainty

• Modeling uncertainty is introduced by considering perturbed mode shapes $\Phi^{(p)}$ for stress estimation:

$$\Phi_i^{(p)} = \sum_{k=-n}^{n} w_k \Phi_i^{(m)}$$

$w_k$ = weights

$\Phi_i^{(m)}$ = i-th mode obtained from the model.

• The Modal Assurance Criterion (MAC) is used to gage the mismatch between model modes and the perturbed modes used for stress prediction:

$$MAC_{i,j} = \left| \frac{\Phi_i^{(p)T} \Phi_j^{(m)}}{\Phi_i^{(p)T} \Phi_i^{(p)}} \cdot \frac{\Phi_j^{(m)T} \Phi_j^{(m)}}{\Phi_j^{(m)T} \Phi_j^{(m)}} \right|^2$$

$i, j = 1, 2, \ldots$
Propagation of uncertainty: Bladed disk
Bladed Disk Characteristics

• The disk is composed of 19 sectors.
• Disk dimensions extrapolated from photographs of a real system.
• Blade geometry representing Purdue Multistage Transonic Compressor [1].

Dimensions
- Blade height: 5.08 cm
- Blade chord: 12.7 cm
- Hub outer radius: 10.16 cm
- Hub inner radius: 1.27 cm
- Disk core thickness: 1.52 cm

Material Properties (Ti-6-4)
- Density: 4430 Kg/m³
- Young’s Modulus: 114 GPa
- Poisson’s Ratio: 0.33

Bladed Disk Finite Element Model

- Blades and disk modeled as one integral component (i.e. a blisk)

- Discretization via SOLID95 elements in the FE commercial package Ansys®

- Externally applied boundary conditions: zero displacement at disk-shaft interface

- Modeling approach accounted for the system’s cyclic-symmetry nature, via internal compatibility constraints
Cyclically Symmetric Structures

- Cyclically symmetric structures analyzed by the modeling of only the fundamental unit (i.e. a sector)

- Equivalent compatibility conditions and loading functions applied to the modeled sector to account for the “missing” part of the structure

- Ansys® employs the *Duplicate Sector Approach*:
  - Fourier transformations used to convert loads applied on any part of the entire structure to equivalent loads applied only on the modeled structure;
  - Existing Ansys® routine has been modified to enable forced harmonic analysis.
### BLADED DISK NATURAL FREQUENCIES [Hz]

<table>
<thead>
<tr>
<th>Mode N.</th>
<th>Harmonic Index</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>1</td>
<td></td>
<td>1070.6</td>
<td>837.45</td>
<td>1188</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>1231.6</td>
<td>837.45</td>
<td>1188</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>1945.3</td>
<td>1606.6</td>
<td>1844</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>3363.8</td>
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<td>1844</td>
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<tr>
<td>5</td>
<td></td>
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<td>3376.1</td>
</tr>
<tr>
<td>Mode N.</td>
<td></td>
<td>3</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td>1412.1</td>
<td>1439.2</td>
<td>1446.8</td>
</tr>
<tr>
<td>2</td>
<td></td>
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<td>1446.8</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3076.3</td>
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<td>3346.8</td>
</tr>
<tr>
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<td>3331.7</td>
<td>3346.8</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>3487.4</td>
<td>4540.6</td>
<td>5409.5</td>
</tr>
</tbody>
</table>

**Harmonic Index = 1; \( \omega_n = 1070.6 \text{ Hz} \)**

**Harmonic Index = 6; \( \omega_n = 3366.4 \text{ Hz} \)**
Bladed Disk: Harmonic Analysis Setup

- Sensor locations: span-wise upper part of blade, suction and pressure surfaces on the same blade

Force applied on blade tip, and of the form:

\[ F_j = F_0 e^{i(\omega t + 2\pi k \frac{(j-1)}{N})} \]

- \( \omega \) = excitation frequency
- \( k \) = harmonic index
- \( j \) = sector number
Uncertainty in sensor’s readings

- Assessment of the impact of sensor measurement uncertainty.

\[
\hat{a}_S(f) = a_S(f) + A \cdot N(0, \sigma^2)
\]

- Excited mode \( \omega_n = 1070.6 \) Hz (k=0).
Uncertainty in sensor’s readings

- Sensor uncertainties in the presence of modeling perturbations

- Excited mode: $\omega_n = 1070.6$ Hz ($k=0$).

$$\mu_{MAC_i} (\sigma_{VM}) - \mu_{MAC_{1}} (\sigma_{VM}) = \begin{cases} 5.73\%, & i = 2 \\ 16.35\%, & i = 3 \end{cases}$$
• Excited mode: @ ω_n = 837.45 Hz (k=1).

• Two sensor measurements needed for stress inference. Forced response near the natural frequency ω_n can be approximated as:

\[ a = \alpha \phi^{(1)}_{\omega_n} + \beta \phi^{(2)}_{\omega_n} \]

where \( \alpha, \beta \in (\mathbb{R}, \mathbb{R}) \)

Two sensor measurements \( (a_{s1}, a_{s2}) \) are needed to compute the coefficients \( \alpha \) and \( \beta \):

\[
\begin{bmatrix}
    a_{s1} \\
    a_{s2}
\end{bmatrix} =
\begin{bmatrix}
    \phi^{(1)}_{s1,\omega_n} & \phi^{(2)}_{s1,\omega_n} \\
    \phi^{(1)}_{s2,\omega_n} & \phi^{(2)}_{s2,\omega_n}
\end{bmatrix}
\begin{bmatrix}
    \alpha \\
    \beta
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
    \bar{\alpha} \\
    \bar{\beta}
\end{bmatrix} =
\begin{bmatrix}
    \phi^{(1)}_{s1,\omega_n} & \phi^{(2)}_{s1,\omega_n} \\
    \phi^{(1)}_{s2,\omega_n} & \phi^{(2)}_{s2,\omega_n}
\end{bmatrix}^{-1}
\begin{bmatrix}
    a_{s1} \\
    a_{s2}
\end{bmatrix}
\]
• Assessment of the impact of sensor measurement uncertainty.

• Excited mode: **double mode** @ \( \omega_n = 3366.4 \text{ Hz} \) (k=6).

\[
\begin{align*}
\begin{bmatrix} a_{s1} \\ a_{s2} \end{bmatrix} &= \begin{bmatrix} \phi_{s1,\omega_n}^{(1)} & \phi_{s1,\omega_n}^{(2)} \\ \phi_{s2,\omega_n}^{(1)} & \phi_{s2,\omega_n}^{(2)} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}
\end{align*}
\]

Impact of measurement uncertainty on maximum-stress estimate

Impact of measurement uncertainty on maximum-displacement estimate
Uncertainty in sensor’s placement

• The position \((x_o, y_o, z_o)\) of a sensor is such that:

\[
z_o = g(x_o, y_o)
\]

where \(g(x,y,\)) is the equation of the blade suction or pressure surface in the xy plane perpendicular to the rotor axis.

• Sensor location uncertainty is simulated by computing random realizations for \((\hat{x}_o, \hat{y}_o)\) such as:

\[
\hat{x}_o = N(x_o, \sigma) \\
\hat{y}_o = N(y_o, \sigma) \\
\hat{z}_o = \text{interpolated value at } (\hat{x}_o, \hat{y}_o)
\]

with: \(\hat{z}_o = g(\hat{x}_o, \hat{y}_o)\)
Effect of mode shape on statistical behavior

- Given a normally-distributed realization \((\hat{x}_o, \hat{y}_o)\), the corresponding mode shapes and curvature may not be normally distributed:
  - Normality is preserved only if local approximation of mode shape and curvature is close to linear;
  - Local shape of the mode used for stress inference affects stress estimations.
Effect of mode shape on statistical behavior

- To evaluate the effect of sensor location on the estimates’ statistical behavior, a total of 18 sensor locations have been considered.
- Sensor nominal positions were normally perturbed with a standard deviation of 0.015” and 5000-run Monte Carlo simulations were performed for each sensor.
Effect of mode shape on statistical behavior

Maximum displacement estimates: distributions on upper surface (Mode at 1070.6 Hz)
Effect of mode shape on statistical behavior

Maximum Von Mises stress estimates: distributions on upper surface (Mode at 1070.6 Hz)
Bayesian Networks
Background: Traditional Uncertainty Propagation

• Uni-directional SISO information flow (Monte Carlo, FORM/SORM, etc.)

• How to deal with more complex situations?
Bayesian (Belief) Networks

• Definition:

A Bayesian network is a data structure used to represent the dependencies among variables from a probabilistic viewpoint. It gives a representation of full joint probability functions.

• Building Blocks:

  – Marginal probabilities for parent (root) nodes
  – Conditional probability tables/distributions, describing relationships of children nodes with respect to their immediate parent nodes (can be obtained by means of parameter learning procedures)
  – These two sets of information are sufficient to fully describe the state space of the system (including all marginal probabilities for children nodes)
  – Additional evidence introduced for children nodes can be used to update marginal distribution at ALL nodes: explicative functionality:
  – Inference of the probability of an event occurring given evidence on related events, that is identification of the most likely scenario explaining the evidence

• Actively used in Artificial Intelligence and Medical Applications. In aerospace applications the use is limited and mainly restricted to failure diagnostics.
Bayesian Networks: Topology

- A Bayesian network is composed of:
  - Variables (nodes); Root nodes are provided with marginal probabilities
  - Causal/Dependence relationships (arrows) quantified by Conditional Probabilities (Tables/Distributions)

<table>
<thead>
<tr>
<th>C</th>
<th>P(S)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRUE</td>
<td>0.1</td>
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<tr>
<td>FALSE</td>
<td>0.5</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C</th>
<th>P(R)</th>
</tr>
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<td>TRUE</td>
<td>0.8</td>
</tr>
<tr>
<td>FALSE</td>
<td>0.2</td>
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</table>

<table>
<thead>
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<th>S</th>
<th>R</th>
<th>P(W)</th>
</tr>
</thead>
<tbody>
<tr>
<td>TRUE</td>
<td>TRUE</td>
<td>0.99</td>
</tr>
<tr>
<td>TRUE</td>
<td>FALSE</td>
<td>0.9</td>
</tr>
<tr>
<td>FALSE</td>
<td>TRUE</td>
<td>0.9</td>
</tr>
<tr>
<td>FALSE</td>
<td>FALSE</td>
<td>0</td>
</tr>
</tbody>
</table>

Example source: Russell S., Norvig P., *Artificial Intelligence A Modern Approach*
Bayesian networks: Numerical Example

- Abductive/explicative inference (use of evidence):
- Given evidence that the grass is wet, which event is more probable, rain or sprinkler being used?

\[
P(R=\text{TRUE} \mid W= \text{TRUE}) = 0.708
\]

\[
P(S=\text{TRUE} \mid W= \text{TRUE}) = 0.430
\]

- Bayesian networks concisely allow to establish the most probable causes associated with a given observation

- If it were also observed that it was raining, the probability of the sprinkler being on would drop to:

\[
P(S=\text{TRUE} \mid W= \text{TRUE}, R=\text{TRUE}) = 0.1945, \text{ as the wet-grass event is explained through rain}
\]
Bayesian-based Uncertainty Propagation: Flow chart

The flowchart illustrates the use of evidence on natural frequencies, where uncertainty propagation is shown for input parameters and modes. Evidence can, in general, be added at any level of the Bayesian network.
Bayesian Updating: 1-input Numerical Example

• The Matlab-based open-source toolbox BNT (by Dr. Kevin Murphy, Univ. of British Columbia) is being used for Bayesian statistical inference.

• As an example, a normally-distributed uncertainty is assumed in the Young’s modulus (E) of a beam with a true value of 114 GPa:

\[ E \sim N(\overline{E}_o, 0.02 \overline{E}_o) \quad \overline{E}_o = 0.9 \times 114 \text{ GPa} \]

• A sample population for the natural frequencies \( f_i \) (i=1,2,3) is then obtained via a Monte Carlo simulation (5000 cases).

• Sample populations for E, f1, f2, and f3 used to train the Bayesian network (parameter learning), where the assumption of linear Gaussian distributions for each child is employed.

• For highly non-linear relationships between children and parent nodes, errors introduced by the linear Gaussian assumption needs to taken into account.
Bayesian Updating: 1-input Numerical Example

- In the case of no evidence:

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>E [GaPa]</td>
<td>102.5860</td>
<td>4.268</td>
</tr>
<tr>
<td>f1 [Hz]</td>
<td>306.62</td>
<td>1.1</td>
</tr>
<tr>
<td>f2 [Hz]</td>
<td>1904.27</td>
<td>4.68</td>
</tr>
<tr>
<td>f3 [Hz]</td>
<td>5256.53</td>
<td>29.02</td>
</tr>
</tbody>
</table>

- For the “true” value of $E_o = 114$ GPa:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>f1 [Hz]</td>
<td>323.25</td>
</tr>
<tr>
<td>f2 [Hz]</td>
<td>2007.5</td>
</tr>
<tr>
<td>f3 [Hz]</td>
<td>5541.5</td>
</tr>
</tbody>
</table>
Beam Example: Inverse Problem (One uncertainty source)

- One random input parameter (Young’s modulus E)
- Up to 10 observations (natural frequencies)
- Bias introduced as $E_{\text{true}} = 90\%$ of 114GPa and $\mu(E) = 114\text{GPa}$
- Net learning with 1000 samples

After 10 observations, the error does not vanish. Bias not completely eliminated.
Beam Example: Inverse Problem (1 uncertainty source) (cont’d)

- One random input parameter (Young’s modulus $E$)
- Up to 10 observations (natural frequencies)
- Bias introduced as $E_{true} = 90\%$ of 114GPa and $\mu(E) = 114$GPa
- Net learning with 1000 samples

Standard deviations are not affected by bias.

Note: evidence introduced from first to last natural frequency.
Beam Example: Inverse Problem (3 uncertainty sources)

- Bias in the form of “true” input values equal to 90% of mean values
- 3 random input parameters (Young’s modulus E, cross section dimensions c and t)
- Net learning with 1000 samples and up to 10 observations (natural frequencies)

Monte Carlo Simulation:

\[ E = N(114 \text{ GPa}, 0.02 \times 114 \text{ GPa}) \]
\[ c = N(15.3 \text{ cm}, 0.05 \times 15.3 \text{ cm}) \]
\[ t = N(2 \text{ cm}, 0.05 \times 2 \text{ cm}) \]
Beam Example: Inverse Problem with Noise

Three net layouts: 1 parent node, 3 parent nodes, and 1 parent node with noise.

Note: evidence added from first to last natural frequency.
Concluding Remarks

• Modeling of simplified sources of uncertainty:
  – input uncertainties;
  – Sensor related uncertainties;
  – Modeling uncertainties

Knowledge gained:
  – Response estimates affected by instrument settings even in simple cases
  – Modeling uncertainties further deteriorates the accuracy of results

• Application of Bayesian Nets for:
  – Integrate of tests results in uncertainty estimation process;
  – Estimate joint probabilities and interactions between sources of uncertainty;
  – Updating.

  – Preliminary application to a beam problem.
Future Work

• Include additional sources of uncertainty at the testing and system modeling level

• Experimental validation of the process:
  – Plate tests have been completed at AEDC.

• Refining BBN procedure to improve direct modeling accuracy (using variable transformations and possibly other than linear Gaussian modeling)

• Application of BBN for updating and uncertainty estimation on bladed disk configuration
Plate test: validating uncertainty models

- Specimen is a 9x4” cantilever plate, with thickness t=0.125”. Gage locations selected with modal results using aluminum.
- Model testing will rely on combinatorial results: i.e., if we have 11 strain gages, and we use 2 to build a model and to compare the results at 9 other locations.
- If we consider a model with Bayesian updating, we also can have potentially significant number of comparisons.

Strain Gage Locations
- Top Surface Z = 0.125
Plate test: validating uncertainty models

- For one material retest with two additional sweep rates, with all other parameters being the same.
  - Recommend steel because of least uncertainty in material properties.

- Also for steel retest with pieces of tape at locations specified based on model to simulate density variations.
  - One additional test should be sufficient
    - Record mass of tape

- Reprocess the current data for the steel plate as follows:
  - Two other different FFT block sizes with all other parameters constant.
  - Two other different windowing functions with other parameters constant.
  - Need to provide the “nominal” DSP parameters used on the tests done to date.

- Test copper plate using previous nominal parameters to represent a material perturbation to the brass.

Strain Gage Locations – Bottom Surface $Z = 0$
Acknowledgments

Work is supported by the Air Force Office of Scientific Research’s (AFOSR) Test & Evaluation Program (Grant # FA9550-05-1-0149)
BACKUP SLIDES
Effect of on sensor location on maximum Von Mises stress estimates’ statistics

Maximum Von Mises stress estimates: distributions on lower surface (Mode at 1070.6 Hz)
Effect of on sensor location on maximum displacement estimates’ statistics

Maximum displacement estimates: distributions on lower surface (Mode at 1070.6 Hz)
Results:
Analysis on Beam-like Blade
Beam-like Blade

System Characteristics

- Length: 22.5 cm
- Thickness: 2 cm
- Chord: 15.3 cm
- Density: 4430 Kg/m³
- Young’s Modulus: 114GPa

Types of uncertainty:

- Analysis Input data uncertainty
  - geometry, material properties.
- Sensor-induced uncertainty
  - placement and measurement inaccuracy.

\[ x_1 = 0.25 \, L_0 \]
\[ x_2 = 0.75 \, L_0 \]
Geometric related Uncertainties

Effect of a variation in blade chord $c_0$ and thickness $t_0$ ($\pm 2\%$ perturbation):

\[
\sigma_a = \max_{\Omega, \omega} \left( \max \left| \sigma_{\text{alternating, axial}} \right| \right)
\]

\[
c = c_0[1 + N(0, \sigma^2)]
\]

\[
t = t_0[1 + N(0, \sigma^2)]
\]

\[3\sigma = .02\]

Some input combinations may cause the structure to critically approach or exceed the fatigue limit.

AFOSR Test & Evaluation Program Review
15-16 August 2006
• Sensors at 25% & 75% of beam length
• Sensor results simulated via harmonic analysis on nominal system
• Sensor noise $A$ defined as a percentage of the maximum amplitude response for the selected mode
• Standard deviation equal to 0.01
MAC matrices for two perturbed modes and original unperturbed mode

Perfect correlation

AFOSR Test & Evaluation Program Review
15-16 August 2006
Reducing modeling uncertainty reduces uncertainty on stress estimation.

**Maximum Stress (output)**

- Unperturbed modes
- Perturbed modes

\[ \sigma_{x}^{(\text{max})} - \mu [\sigma_{x}^{(\text{max})}] \text{ [Pa]} \]
Reducing modeling uncertainty reduces uncertainty on stress estimation.

### Graph

**Maximum Stress (output)**

- **Initial Model**
- **Improved (perfect) model**
- **Input uncertainty**
- **Modeling uncertainty**

**Equation:**

\[
\sigma_x^{(\text{max})} - \mu[\sigma_x^{(\text{max})}] \text{ [Pa]}
\]
Bladed Disk: Effect of Modeling Uncertainty

- Combination of modeling and sensor-induced uncertainties.

- Excited mode: double mode @ $\omega_n = 3366.4$ Hz ($k=6$).

- Both sensor readings perturbed.

\[ \mu_{MAC_i}(\sigma_{VM}) - \mu_{MAC_1}(\sigma_{VM}) = \begin{cases} -2.37\%, i = 2 \\ -4.67\%, i = 3 \end{cases} \]
Back-up Slides
BBN
Bayesian Networks: Bayes’ Theorem

Bayes’ theorem represents the fundamental idea behind Bayesian networks. In the simplest case of two Boolean events A and E, the theorem establishes that:

\[
P(A \mid E) = \frac{P(E \mid A)P(A)}{P(E)}
\]

- \(P(A \mid E)\) = posterior probability (probability of A given evidence \(E\))
- \(P(E \mid A)\) = likelihood
- \(P(A)\) = prior probability
- \(P(E)\) = probability of data (evidence)

Therefore, given evidence for event E, the probability associated with event A can be updated appropriately.
Bayesian Updating: Flowchart

Uncertainties on Nominal Model Parameters
- Geometry Dimension L
- Material density \( \rho \)
- ... (omitted)
- Young’s Modulus E

Monte Carlo Simulation

Modal Analysis on System Model

Modal Analysis Results

Experiment on Physical System:
- Measured Mode \( Z_i \)
- Measured frequency \( \lambda_i \)

Bayesian Updating

Repeat as more evidence becomes available
Beam Example: Net Training (1 parent node)

- Training causes a change in the statistical parameters of the prior/marginal probabilities with respect to the parameters associated with the training data (Monte Carlo simulation).

- For the no-bias case:
  - the mean $\mu$ does not vary considerably;
  - the standard deviation $\sigma$ changes.
Beam Example: Change in Prior Probability (1 parent node)

- Bayesian net trained using sample data from a Monte Carlo simulation with $E \sim N(0.9 \cdot E_{true}, 2\% \cdot 0.9 \cdot E_{true})$.
- Then, prior probability of $E$ modified based on another Monte Carlo simulation with $E \sim N(E_{true}, 2\% \cdot E_{true})$.

Despite a bias on $E$ in the training data, the adjusted prior probability reduces the error in the marginal probabilities of all network nodes.
Beam Example: Net Training (3 parent nodes)

- Training causes a change in the statistical parameters of the prior/marginal probabilities with respect to the parameters associated with the training data (Monte Carlo simulation).

- No-bias case:
  - the mean $\mu$ does not vary considerably;
  - the standard deviation $\sigma$ changes.
Beam Example: Inverse Problem (1 uncertainty source)

- 1 random parameter (Young's modulus E)
- Up to 10 observations (natural frequencies).
- No bias introduced.
- Net learning with 1000 samples vs. 5000 samples.

Monte Carlo Simulation:
\[ E = N(114 \text{ GPa}, 0.02 \times 114 \text{ GPa}) \]

The number of samples in net learning has a marginal effect in this circumstance.
Beam Example: Direct Problem (1 uncertainty source)

- Input training data: \( E \sim N(0.9 \cdot E_{\text{true}}, 2\% \cdot 0.9 \cdot E_{\text{true}}) \)

- The bias in the training data is embedded both \( E \) and \( f_i \). Therefore, a correct evidence \( E \) produces \( \mu(f_i) \) close to the true natural frequencies (scaling effect).

![Diagram showing direct propagation with evidence E](image)
Beam Example: Inverse Problem (3 uncertainty sources)

- 3 random input parameters (Young’s modulus E, cross section dimensions c and t).
- Up to 10 observations (natural frequencies).
- Net learning with 1000 samples vs. 5000 samples with no bias.
- Bias present: “true” input values equal to 90% of mean values.

Monte Carlo Simulation:

\[ E = N(114\text{GPa}, 0.02*114\text{GPa}) \]
\[ c = N(15.3\text{cm}, 0.05*15.3\text{cm}) \]
\[ t = N(2\text{cm}, 0.05*2\text{cm}) \]