HOMOGENIZATION OF VIBRATING PERIODIC STRUCTURES

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Objectives and motivations

OBJECTIVES:

Derivation of continuum equations for periodic lattice structures

Approximation of dynamic behavior and wave propagation characteristics (Harmonic response and dispersion relations)

Formulation of macro-elements for the numerical solution of continuum equations

MOTIVATIONS:

Full description of the dynamic behavior may be computationally intensive (large number of elements)

Continuum equations may provide insight in the “equivalent” mechanical properties of periodic media
Periodic assemblies of engineering interest

Material-like WOVEN composites

CHIRAL honeycomb with NEGATIVE Poisson’s ratio

FOAMS featuring stochastic architecture

Outline

Definition of periodic assemblies

Fourier Analysis of discrete periodic media

Homogenization:
  Multiscale formulation
  Derivation of continuum equations

Example 1: one dimensional system

Example 2: Two-dimensional system

Conclusions
PERIODIC STRUCTURE: STRUCTURE FEATURING A REPETITIVE ELEMENT KNOWN AS ELEMENTARY UNIT CELL

EXAMPLES OF PERIODIC_STRUCTURES

One-dimensional bi-material bar

One-dimensional truss structure
Fourier analysis for discrete media

Elasto-dynamic equations
\[ \mathcal{D}[u(m)] = f(m) \]

Cell’s degrees of freedom
\[ u(m) \]

Locations of nodes within the cell

Discrete spatial Fourier Transform
\[ \hat{u}(m) = \sum_m e^{im \cdot \xi} \cdot u(m) \quad \xi \in (-\pi, \pi)^d \quad m \in \mathbb{Z}^d \]

Applying Transformation to dynamic equations
\[ \Gamma(\xi) \hat{u}(\xi) = \hat{f}(\xi) \]

“SYMBOL” OF THE SYSTEM
\[ \Gamma(\xi) = \sum_n e^{in \cdot \xi} D_n \]

Solving the linear system and evaluating the Inverse Fourier Transform yields
\[ u(m) = \frac{1}{(2\pi)^d} \int_{(-\pi, \pi)^d} \Gamma(\xi)^{-1} \hat{f}(\xi) \cdot e^{-in \cdot \xi} d\xi \]
**HOMOGENIZATION**: technique to derive equivalent continuum equations that approximate the behavior of a given periodic structure

**CONCEPT OF DIFFERENT LENGTH SCALES** between the macroscopic structure and the repetitive element is captured through the introduction of parameter $\varepsilon$ such that

$$\varepsilon = \frac{x}{y}$$

- $x =$ characteristic dimension of the repetitive element
- $y =$ characteristic dimension of the macroscopic system
Homogenization of lattice equations

Scaled lattice equations become

\[ \mathcal{D}^{(\varepsilon)}[u^{(\varepsilon)}(m)] = \tilde{f}^{(\varepsilon)}(m) \]

It’s shown in the literature that the symbol can be scaled like:

\[ \Gamma^{(\varepsilon)}(\xi) = \varepsilon^2 \Gamma^{(\varepsilon \xi)} \]

With the scaled symbol the solution in the Fourier domain becomes

\[ \hat{u}^{(\varepsilon)}(\xi) = [\Gamma^{(\varepsilon)}(\xi)]^{-1} \hat{f}^{(\varepsilon)}(\xi) \]

A Taylor expansion of the symbol can be performed such that

\[ S^{(\varepsilon,p)}(\xi) = [\Gamma^{(\varepsilon)}(\xi)]^{-1} + \mathcal{O}(\varepsilon^{2p+2}) \]

Recalling the use of the Inverse Fourier Transform, a continuous variable

\[ u^{(\varepsilon,p)} = \mathcal{F}^{-1}[S^{(\varepsilon,p)} \hat{f}(\varepsilon)] \]
One dimensional spring-mass system

\[ \Gamma(\xi) = -m\omega^2 + [2k(1 - \cos\xi)] \]

\[ \omega^2 = \frac{1}{m} [2k(1 - \cos\xi)] \]

**Exact Dispersion Relations**

**Approximate Dispersion Relations**

**I Order**

\[ \omega^2 = \frac{1}{m} \left[ k\xi^2 \right] \]

**II Order**

\[ \omega^2 = \frac{1}{m} \left[ k(\xi^2 - \frac{\varepsilon^2\xi^4}{12}) \right] \]

**III Order**

\[ \omega^2 = \frac{1}{m} \left[ k(\xi^2 - \frac{\varepsilon^2\xi^4}{12} + \frac{\varepsilon^4\xi^6}{360}) \right] \]

**Progressive Taylor Series of Trigonometric Quantities**
One dimensional spring-mass system

Low order approximations work only in the low-wave-number limit
Increasing the order yield increasing agreement with the exact curve
One-dimensional bi-material bar

Homogenized equation

\[ [a_0(\omega) + \sum_{j=1}^{p} a_{2j}(\omega) \varepsilon^{2j-2} \frac{d^{2j}}{dx^{2j}}] u^{(\varepsilon,p)}(x) = 0 \]

TRUNCATION AT FIRST ORDER (p=1)

\[ a_2(\omega) \frac{d^2 u^{(\varepsilon,1)}(x)}{dx^2} + a_0(\omega) u^{(\varepsilon,1)}(x) = 0 \]

where

\[ a_0(\omega) = \rho_h A \omega^2 \]
\[ a_2(\omega) = E_h A \]
\[ \rho_h = (1 - \alpha) \rho_1 + \alpha \rho_2 \]

\[ E_h = \frac{E_1 E_2}{\alpha E_1 + (1 - \alpha) E_2} \]

Same results Using the Rule of Mixtures
One-dimensional bi-material bar

O.D.E.

\[ a_2(\omega) \frac{d^2 u(x)}{dx^2} + a_0(\omega) u(x) = 0 \]

BOUNDARY CONDITIONS

\[ u(x = 0) = 0 \]
\[ a_2(\omega) \frac{d u(x)}{dx} \bigg|_{x=L} = -F \]

DEFINITION OF PROPER BOUNDARY CONDITIONS
MAJOR LIMITATION TO HIGHER ORDER PDEs
Two-dimensional structure

Simply supported 2D Spring-mass system

REPETITIVE ELEMENT

\[ k_4 \quad k_3 \quad k_2 \quad k_1 \]

\[ m \]

Assumption:
OUT-OF-PLANE DISPLACEMENTS

Simply supported along the contour
Derivation of continuum equations

CONTINUUM EQUIVALENT PDE TRUNCATED AT FIRST ORDER

\[(k_1 + k_2 + k_4) \frac{\partial^2 w(x,y)}{\partial x^2} + (k_3 + k_2 + k_4) \frac{\partial^2 w(x,y)}{\partial y^2} + \]
\[+ 2(k_2 - k_4) \frac{\partial^2 w(x,y)}{\partial x \partial y} + \omega^2 mw(x,y) = f(x,y)\]

Special case:

\[k \nabla^2 w(x,y) + \omega^2 mw(x,y) = f(x,y)\]
\[\nabla^2 w + cw = f\]

DYNAMIC EQUATION OF A VIBRATING MEMBRANE

Approximate solution found though a Finite Element Formulation for the weak statement of the continuum PDE

FE analysis conducted discretizing the bi-dimensional domain into rectangular “super” elements
Analysis of dispersion relations

PHASE CONSTANT SURFACES

Considered structure

\[ \xi_2(\varepsilon) \]

I order
Exact vs. \( [O(\varepsilon^2)] \)

\[ \xi_1(\varepsilon) \]

III order
Exact vs. \( [O(\varepsilon^6)] \)

\[ \xi_2(\varepsilon) \]

II order
Exact vs. \( [O(\varepsilon^4)] \)

\[ \xi_1(\varepsilon) \]

IV order
Exact vs. \( [O(\varepsilon^8)] \)
Response to harmonic load

FRF at the mid point

Considered structure

![Graph showing response to harmonic load with FRF at the mid point and considered structure. The graph includes lines for discrete structure and continuum equivalent, labeled with FEM.](image)
Refinement of the approximation

**ORIGINAL APPROXIMATE SYMBOL**

\[ \Gamma^\varepsilon(\omega) = (k_1 + k_2 + k_4)\xi_1^2 + (k_3 + k_2 + k_4)\xi_2^2 + 2(k_2 - k_4)\xi_1\xi_2 - m\omega^2 \]

**IMPOSED “FITTING” SYMBOL**

\[ \Gamma^\varepsilon(\omega) = a_1(\omega)\xi_1^2 + a_2(\omega)\xi_2^2 + a_3(\omega)\xi_1\xi_2 - m\omega^2 \]

\(a_1, a_2, a_3\) are determined through least square method in order to FIT the EXACT dispersion relations.
Refinement of the approximation

Standard homogenization

Refined approximation through least squares fit
Refinement of the approximation

BI-MATERIAL BAR

Standard homogenization

Refined approximation through least squares fit

\[ \frac{u(L)}{F} \]

\[ \omega \text{ [rad/s]} \]

\[ \omega \text{ [rad/s]} \]
Multi-field analysis

SINGLE-FIELD APPROACH

\[ q_a = \frac{1}{2} (q_L + q_I) = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} q_L \\ q_I \end{bmatrix} \]

\[ q_d = \frac{1}{2} (q_I - q_L) = \frac{1}{2} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_L \\ q_I \end{bmatrix} \]

NEW SETS OF DEGREES OF FREEDOM

Obtained through a change of variables

The Average is related to average properties of the whole structure.
It approximates the behavior of the structure in the long wavelength limit.

The semi-difference is related to the small differences/changes within the unit cell.
It approximates the behavior of the structure in the short wavelength limit.
Two-field analysis - Results

TRUSS STRUCTURE

\[ E, \rho, A \]

\[ \xi [1/m] \]

\[ \omega [\text{rad/s}] \]

Exact

Branch I

Branch II

Graphs showing the relationship between \( \xi [1/m] \) and \( \omega [\text{rad/s}] \) for different branches.
Conclusions and future work

Conclusive Remarks

THE HOMOGENIZATION PROCEDURE PROVIDES A WAY TO OBTAIN APPROXIMATE DISPERSION RELATIONS AND HARMONIC RESPONSE FOR A LATTICE

REFINEMENT OF THE APPROXIMATION CAN BE ACHIEVED THROUGH INTERPOLATION OF THE COEFFICIENTS OF THE PDE TO FIT THE EXACT DISPERSION RELATIONS

Current and future work

ATTEMPT TO MAP A CORRESPONDENCE BETWEEN A GIVEN LATTICE STRUCTURE AND THE CORRESPONDENT FAMILY OF EQUIVALENT PDEs